<u>upperbound</u>: $S \subseteq \mathbb{R}$. Then $\alpha \in \mathbb{R}$ is an upperbound if $\alpha \geq x \ \forall x \in S$.

~~scopa~~

<u>Lower bound</u>: $S \subseteq \mathbb{R}$. Then $\beta \in \mathbb{R}$ is a lower bound if $\beta \leq x, \forall x \in S$.

clearly if S has an upper bound $\alpha \in \mathbb{R}$ then $\alpha+1, \alpha+2, \ldots$ all are upper bounds of S.
Similarly if β is lower bound then $\beta-1, \beta-2, \ldots$ all are lower bounds.

<u>Bounded above set</u>: A set $S \subseteq \mathbb{R}$ is called bounded above if S has a upper bound.
<u>Bounded below set</u>: A set $S \subseteq \mathbb{R}$ is called bounded below if S has lower bound.
<u>Bounded set</u>: A set $S \subseteq \mathbb{R}$ is called bounded if S is bounded above set and bounded below set. i.e, if S is bounded above as well as bounded below.

<u>Ex</u>: Let $S = \{x \in \mathbb{R} : x^2 < 1\}$ clearly $\forall x \in S$
$x^2 < 1 \Rightarrow x < x^2 < 1$ so $\forall x \in S, x < 1$
Therefore S is bounded above set. and bounded by 1.

<u>Note</u>: From above example 1 is upper bound of S then 2, 3, 2.5, 3.5, ... all real numbers greater than 1 are upper bound.
Let $U = \{$ set of all upper bound of $A\}$
$= \{1, 1.2, 1.3, \ldots 2, \ldots \}$
$= \{x \in \mathbb{R} : x \geq 1\}$
clearly we can say about least element of U

but not about greatest element. From this we introduce the concept of <u>least upper bound</u>. Similarly for bounded below set $A = \{x \in \mathbb{R} : x \geq 1/2\}$, we see:

$U' = \{$ set of all lower bound of $A\}$

$= \{1/2, 1/3, 1/4, 0, -1, \ldots\}$

$= \{x \in \mathbb{R} : x \leq 1/2\}$.

So we can say about greatest element of U' but not lowest element. From this we introduce the concept of greatest lower bound.

First we say existance of this two concept:

<u>least upper bound</u>: (lub) An upper bound α of $S \subseteq \mathbb{R}$ called lub if it is the least upper bound. i.e. α is less than every upper bound / which is also called supremum.

<u>Greatest lower bound</u>: (glb / infimum).

A ~~upper~~ lower bound β of $S \subseteq \mathbb{R}$ called infimum / glb if β is grater than all other lower bound.

<u>lub Axiom</u>: A least upper bound exists for any $S \neq \phi$, $S \subseteq \mathbb{R}$ which is bounded ~~abounded~~ above.

<u>glb Axiom</u>: A greater upper bounded exists for any $S \neq \phi$, $S \subseteq \mathbb{R}$ which is bounded below.

For a non-empty set S, bounded above, the supremum of S is denoted by $\boxed{\sup S}$, also $\sup S$ may or may not belongs to S.

Also infimum of S is denoted by $\boxed{\inf S}$ which may or may not belongs to S.

Scanned by CamScanner

If S is finite then

(i) $\sup S = \max S$

(ii) $\inf S = \min S$.

Th: (Least upper bound axiom).
Every non-empty Subset of $\mathbb{R}$ that is bounded above has a supremum.

proof: Let $S (\neq \emptyset) \subseteq \mathbb{R}$. and S is bounded above. So $\exists\, \alpha \in \mathbb{R}$ s.t $x \leq \alpha \ \forall x \in S$.

Let $U = \{u \in \mathbb{R} : u \text{ is a upper bound of } S\}$.

clearly $U \neq \phi$, ($\because$ S is bounded above, $\alpha \geq x \ \forall x \in S$ so $\alpha \in U$].

Again $u \in U$ and $x \in S \Rightarrow x \leq u$

$\Rightarrow U$ is bounded below.

$\Rightarrow U (\neq \emptyset) \subseteq \mathbb{R}$ and bounded below

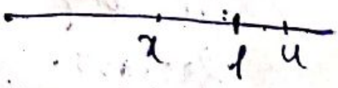
Then by "infimum axiom" U has infimum.

Let $l = \inf U$. Then $\forall u \in U \Rightarrow l \leq u$

Since every $x \in S$ is lowerbound of U and
$l = \inf U \Rightarrow l \geq x \ \forall x \in S$.

So l is upperbound of S and $u \geq l \ \forall u \in U$.

Hence $\boxed{l = \sup S}$.



Th: (Greatest lower bound axiom)
Every non-empty Subset of $\mathbb{R}$ that is bounded below has a infimum (H.T).

Th: Let S be a non-empty subset of $\mathbb{R}$, bounded below. A lower bound l of S is infimum of S iff for each $\varepsilon > 0$ $\exists$ $x \in S$ s.t (depend on ε)

$l \le x < l + \varepsilon$

$\Rightarrow$ proof:

Let $l = \inf S$. Let $\varepsilon > 0$. Since l is infimum of S then l is greatest lower bound of S.

Since l is lower bound so $\forall x \in S$, $l \le x$.

Now $l + \varepsilon > l$, if $l+\varepsilon$ is lower bound then l can not be come greater which is contradiction. so $l+\varepsilon$ is not lower bound so $\exists$ $x \in S$ s.t $x < l+\varepsilon$. Therefore $l \le x < l+\varepsilon$.

Conversely let for each $\varepsilon > 0$ $\exists$ $x \in S$ s.t

$l \le x < l+\varepsilon$.

we shall show that l is infimum of S.

To show this we first show l is lower bound and after then it is greatest.

it is given that l is lower bound. For greatestness of l.

Let l_0 is lower bound and $l < l_0$

Then $l_0 - l > 0 \Rightarrow \frac{l_0 - l}{2} > 0$ Let $\varepsilon = \frac{l_0 - l}{2}$

Then by given condition $\exists$ $x \in S$ s.t

$l \le x < l + \varepsilon$.

$\Rightarrow x < l + \varepsilon$

$\Rightarrow x < l + \frac{l_0 - l}{2}$

$\Rightarrow x < \frac{l_0 + l}{2}$

$\Rightarrow x < l_0 - \varepsilon < l_0$

From $\varepsilon = \frac{l_0 - l}{2}$

$\Rightarrow 2\varepsilon = l_0 - l$

$\Rightarrow \varepsilon + l = l_0 - \varepsilon$.

$\Rightarrow$ But l_0 is lowerbound of S i.e.
$\forall x \in S$, $l_0 \leq x$.

So l_0 is not lower bound. Hence
$\therefore$ ϕ is greatest lowerbound.

Properties of Supremum and infimum

(i) $S (\neq \phi) \subseteq \mathbb{R}$, and $L = \sup S$

$\Rightarrow$ (a) $x \leq L$ $\forall x \in S$
(b) for any $\varepsilon > 0$ $\exists$ $x_0(\varepsilon) \in S$ s.t
$L - \varepsilon < x_0 \leq L$

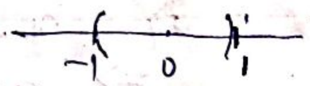
(ii) $S (\neq \phi) \subseteq \mathbb{R}$ and $l = \inf S$

$\Rightarrow$ (a) $l \leq x$ $\forall x \in S$
(b) for any $\varepsilon > 0$ $\exists$ $x_0(\varepsilon) \in S$ s.t
$l \leq x_0 < l + \varepsilon$

problem:

(i) worked Example (S. K. mapa) (H.T)
(ii) Find Supremum and infimum of $A = (0,1)$.
(iii) Find Sup A and inf A where
(a) $A = \{ x \in \mathbb{R} : x^2 < 1 \}$.
(b) $A = \{ x \in \mathbb{R} : 3x^2 + 8x - 3 < 0 \}$
(c) $A = \{ \frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N} \}$
(d) $A = \{ \frac{n + (-1)^n}{n} ; n \in \mathbb{N} \}$.

$\Rightarrow$ (a) $A = \{ x \in \mathbb{R} : x^2 < 1 \}$.



if $x = 1$ then $x^2 = 1$
so $1 \notin A$
if $x = -1$ $x^2 = 1$ $x = -1 \notin A$
every element $x > 1$ $\notin A$
$x < -1 \notin A$